$$u_k(x, 0) = u(x, t_k) = \varphi_k(x)$$

which contradicts the previous assumption, Q. E. D.

In conclusion thanks are given to G. I. Barenblatt and V. M. Entov for drawing the author's attention to this problem.

BIBLIOGRAPHY

1. Barenblatt, G. I. and Krylov, A. P., On the elastoplastic mode of filtration. Izv. Akad. Nauk SSSR, OTN, №2, 1955.
2. Barenblatt, G. I., On certain problems concerning the reduction of pressure and the propagation of the pressure relief wave, in the elastoplastic mode of filtration. Izv. Akad. Nauk SSSR, OTN, №2.
3. Kamenomostskaia, S. L., On a problem of the theory of filtration. Dokl. Akad. Nauk SSSR, Vol. 116, №1, 1957.
4. Il'in, A. M., Kalashnikov, A. S. and Oleinik, O. A., Second order, parabolic, linear equations. Usp. matem. n., Vol. 17, №3, 1962.

Translated by L. K.

UNSTEADY WAVES IN A ROTATING CHANNEL OF CONSTANT DEPTH

PMM Vol. 33, №6, 1969, pp. 1085-1090
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(Received August 1, 1969)

Propagation of waves caused by the original rise on the surface of rotating liquid is considered. The deformation of the disturbed level proceeds in accordance with the theory of long waves. The unsteady part of the wave rise may be treated as a limit superposition of standing waves with phases allowing for a complete range of wave numbers. The original perturbations are assumed to act in such a manner that, as the distance between the nodes decreases, the elementary crests of an arbitrary component take a near-equilibrium position. It is permissible to use in such analysis of unsteady wave problems a Fourier integral the complex amplitude of which must be determined. Our analysis of waves in a channel is based on Sretenskii's general hydrodynamic analysis of tsunami waves on a rotating half-plane [1].

1. The value of the perturbed level in a channel of constant depth h can be found from the wave equation

$$\frac{\partial^2\zeta}{\partial t^2} + 4\omega^2\zeta = gh\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) \tag{1.1}$$

in which the initial functions are

$$\zeta(x, y, 0) = M(x, y), \quad \partial\zeta(x, y, 0)\,/\,\partial t = N(x, y) \tag{1.2}$$

those perturbed motions that do not result variations of the liquid level in time are not considered.

Assuming shockless initial values of the transverse components of the velocity and acceleration, the conditions of impenetrability at the boundaries $y = 0$, $y = l$ can be written as follows:

$$\frac{\partial^2 \zeta}{\partial t\, \partial y} - 2\omega \frac{\partial \zeta}{\partial x} = 0 \qquad (1.3)$$

which indicates that the Coriolis parameter is present.

The initial perturbations (1.2) on an open surface, subjected to a centrifugal inertial force, will be reflected repeatedly when meeting perpendicular walls of the channel. The problem consists in finding the form of the free surface in the channel covered by unsteady waves.

2. The wave rise $\zeta(x, y, t)$ is found using the integral

$$\zeta(x, y, t) = \int_{-\infty}^{\infty} A(t, y, k) e^{ikx}\, dk \qquad (2.1)$$

in which the complex amplitude $A(t, y, k)$ is not known. According to the inverse Fourier transformation the initial functions (1.2) can be represented as the following integrals:

$$f(y, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(\xi, y) e^{-ik\xi}\, d\xi, \quad \varphi(y, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\xi, y) e^{-ik\xi}\, d\xi$$

When (2.1) is substituted into (1.1), we have the well-known telegraphy equation

$$\frac{\partial^2 A}{\partial t^2} - c^2 \frac{\partial^2 A}{\partial y^2} + s^2 A = 0, \qquad s^2 = 4\omega^2 + c^2 k^2 \qquad (2.2)$$

with the transformed initial conditions

$$A(0, y, k) = f(y, k), \qquad \frac{\partial A(0, y, k)}{\partial t} = \varphi(y, k) \qquad (2.3)$$

The boundary conditions of reflection from the channel walls $y = 0$, $y = l$ follow from the state of impenetrability; they are

$$\partial^2 A \,/\, \partial t \partial y - 2i\omega k A = 0 \qquad (2.4)$$

The problem of finding $A(t, y, k)$ can be reduced to integrating a nonhomogeneous hyperbolic equation

$$\partial^2 A \,/\, \partial t \partial y - 2i\omega k A = B(t, y, k) \qquad (2.5)$$

For the function $B(t, y, k)$, which is the right side of (2.5), the problem may now be formulated as follows: a function $B(t, y, k)$ must be found that would satisfy the homogeneous equation

$$\frac{\partial^2 B}{\partial t^2} - c^2 \frac{\partial^2 B}{\partial y^2} + s^2 B = 0 \qquad (2.6)$$

and the homogeneous boundary conditions

$$B(t, 0, k) = B(t, l, k) = 0 \qquad (2.7)$$

and would have Cauchy initial conditions. i.e.

$$B(0, y, k) = m(y, k), \qquad \frac{\partial B(0, y, k)}{\partial t} = n(y, k) \qquad (2.8)$$

where known functions $m(y, k)$ and $n(y, k)$ are related to the quantities in (2.3) by means of the following expressions:

$$m(y, k) = \frac{\partial \varphi}{\partial y} - 2i\omega k f(y, k), \quad n(y, k) = c^2 \frac{\partial^3 f}{\partial y^3} - s^2 \frac{\partial f}{\partial y} - 2i\omega k\, \varphi(y, k) \qquad (2.9)$$

3. Within the band $t \geqslant 0$, $0 \leqslant y \leqslant l$ the solution of the characteristic problem of Cauchy with the mixed boundary conditions (2.7), (2.8) can be found using the Hadamard method [2]. The two-parameter family of characteristics $y \mp ct = \text{const}$ splits the region in which the function $B(t, y, k)$ exists into a number of separate subregions;

an analog of the Riemann function, with a constant discontinuous change at the point in which a characteristic is crossed, is determined for each subregion. Thus, when $ct \leqslant y$, the Riemann function is of the form

$$R(\sigma) = J_0((s/c)\sqrt{\sigma}) = J_0((s/c)\sqrt{c^2t^2 - (y-\eta)^2})$$

which corresponds to the differential equation (2.6) without the effect of the boundaries. Up to a certain instant, $B(t, y, k)$ represents free propagation of the disturbances from their sources and can be written as

$$B(t, y, k) = {}^1/_2[m(y-ct, k) + m(y+ct, k)] + \frac{1}{2c}\int_{y-ct}^{y+ct}[R(\sigma)n(\eta, k) + 2c^2t R'(\sigma)m(\eta, k)]\,d\eta \tag{3.1}$$

The representation in the form of travelling waves (3.1) makes it possible to confirm that the initial conditions (2.8) are satisfied. The boundary condition at the wall $y=0$ will be satisfied if the Riemann function is built by the mapping method

$$R(\sigma_1) = J_0((s/c)\sqrt{\sigma_1}) = J_0((s/c)\sqrt{c^2t^2 - (y+\eta)^2})$$

and then, for $ct \geqslant y$,

$$B(t, y, k) = {}^1/_2[m(ct+y, k) - m(ct-y, k)] + \frac{1}{2c}\int_0^{ct+y}[R(\sigma)n(\eta, k) + 2c^2t R'(\sigma)m(\eta, k)]\,d\eta - \frac{1}{2c}\int_0^{ct-y}[R(\sigma_1)n(\eta, k) + 2c^2t R'(\sigma_1)m(\eta, k)]\,d\eta \tag{3.2}$$

The two forms (3.1) and (3.2) are equivalent for an odd continuation of the initial functions (2.9) across the boundary $y = 0$.

In the general case, for the purpose of satisfying the requirements of wave reflection (2.7) it is necessary to place fictitious sources in consecutive order across the band at the points $2q_1l + \eta$, $\eta - 2q_2l$, $-\eta - 2q_1'l$, $2q_2'l - \eta$

after a suitable regrouping the common effect of these sources will be represented by the function of Riemann-Hadamard discontinuous solution

$$H(t, y, \eta, k) = \sum_{q_1} R(\sigma_{q_1}) + \sum_{q_2} R(\sigma_{q_2}) - \sum_{q_1'} R(\sigma_{q_1'}) - \sum_{q_2'} R(\sigma_{q_2'}) \tag{3.3}$$

In a circle with a radius ct the quantities

$$\sigma_{q_1} = c^2t^2 - (y - \eta - 2q_1l)^2, \quad \sigma_{q_2} = c^2t^2 - (y - \eta + 2q_2l)^2$$
$$\sigma_{q_1'} = c^2t^2 - (y + \eta + 2q_1'l)^2, \quad \sigma_{q_2'} = c^2t^2 - (y + \eta - 2q_2'l)^2 \tag{3.4}$$

are the squares of the semichords passing through the point sources perpendicularly to the central axis. The geometry makes it clear that there are no reflected waves at any given instant, if the entire periphery of the circle is within the band $t \geqslant 0$, $0 \leqslant y \leqslant l$. The consecutive instants of reflection are associated with integers limited by the inequalities

$$\frac{y+ct-l}{2l} \leqslant q_1 \leqslant \frac{y+ct}{2l}, \qquad \frac{ct-y}{2l} \leqslant q_2 \leqslant \frac{ct-y+l}{2l}$$
$$\frac{ct-y-l}{2l} \leqslant q_1' \leqslant \frac{ct-y}{2l}, \qquad \frac{ct+y}{2l} \leqslant q_2' \leqslant \frac{ct+y+l}{2l} \tag{3.5}$$

The number of summands in each of the sums in (3.3) may be either the same or may differ by unity, depending on the time and the location at which the arrival of the wave is observed. The lower and upper values of the inequalities (3.5) indicate the order in which individual perturbations arrive at a given point, including the channel boundaries.

A term-by-term application of the Green formula in the regions bounded by the sections of the boundaries and by the sections of adjacent characteristics leads to the following expression:

$$
\begin{aligned}
B(t, y, k) = {} & \frac{1}{2} \sum_{q_1} m(y + ct - 2q_1 l, k) + \frac{1}{2} \sum_{q_2} m(y - ct + 2q_2 l, k) + \\
& + \frac{1}{2} \sum_{q_1'} m(y - ct + 2q_1' l, k) + \frac{1}{2} \sum_{q_2'} m(y + ct - 2q_2' l, k) + \\
& + \frac{1}{2c} \sum_{q_1} \int_0^{\eta_1} [R(\sigma_{q_1}) n(\eta, k) + 2c^2 t R'(\sigma_{q_1}) m(\eta, k)] d\eta + \\
& \qquad (\eta_1 = y + ct - 2q_1 l) \\
& + \frac{1}{2c} \sum_{q_2} \int_{\eta_2}^{l} [R(\sigma_{q_2}) n(\eta, k) + 2c^2 t R'(\sigma_{q_2}) m(\eta, k)] d\eta - \\
& \qquad (\eta_2 = 2q_2 l + y - ct) \\
& - \frac{1}{2c} \sum_{q_1'} \int_0^{\eta_3} [R(\sigma_{q_1'}) n(\eta, k) + 2c^2 t R'(\sigma_{q_1'}) m(\eta, k)] d\eta - \\
& \qquad (\eta_3 = ct - y - 2q_1' l) \\
& - \frac{1}{2c} \sum_{q_2'} \int_{\eta_4}^{l} [R(\sigma_{q_2'}) n(\eta, k) + 2c^2 t R'(\sigma_{q_2'}) m(\eta, k)] d\eta \\
& \qquad (\eta_4 = 2q_2' l - y - ct)
\end{aligned}
\tag{3.6}
$$

the prime in the function R denotes differentiation with respect to any transformation argument (3.4). Our final result (3.6) is a finite combination of individual solutions of two kinds: in the subregion which includes the limit of the band, the solutions correspond to the boundary value problem with the Cauchy initial data; between the sections of characteristics, the solutions can be obtained by means of Riemann function. The expression (3.6) which involves imaginary sources is the generalization of the expression (3.2), for an arbitrary instant. Thus, the ancillary problem of finding the function $B(t, y, k)$ in Eq. (2.5) is now solved.

4. In the characteristic Cauchy problem for Eq. (2.5) with the conditions (2.3) there are not enough data on the characteristic $y = 0$. Indeed, conditions (2.5) are given only on the characteristic $t = 0$ and they cannot, therefore, determine the integral of the hyperbolic equation (2.5). The condition (2.3) are compatible with (2.8) and do not contradict Eq. (2.5), but they do not introduce any new elements in our problem either. For a complete determination of the integral it is assumed that function A has along the characteristic $y = 0$ the values of a function $F(t, k)$ which can be found from the supplementary postulate [1]. In our case the integral of Eq. (2.5), in accordance with the data in the characteristics $A(0, y, k) = f(y, k)$, $A(t, 0, k) = F(t, k)$, must everywhere satisfy the telegraphy equation (2.2), including the boundary $y = 0$; $F(t, k)$ is assumed continuous, and having in the point $t = y = 0$ a single-valued derivative, calculated for various directions.

The function $A(t, y, k)$ is developed by the Riemann function method

$$A(t, y, k) = F(t, k) + f(y, k) - f(0, k) S(\rho_0) + t \int_0^y f(\eta, k) S'(\rho_1) d\eta +$$
$$+ y \int_0^t F(\tau, k) S'(\rho_2) d\tau + \int_0^t d\tau \int_0^y B(\tau, \eta, k) d\eta \quad (4.1)$$

We have preserved in the above the original notation of Sretenskii [1]: $S(\rho)$ is the Riemann function for which

$S(0) = 1,\ \rho = (t - \tau)\ (y - \eta), \quad \rho_1 = t(y - \eta), \quad \rho_2 = y(t - \tau),\ \rho_0 = ty$

The left side of (2.2) becomes on the characteristic $y = 0$ an integro-differential equation

$$F''(t, k) + s^2 F(t, k) + 4\omega^2 c^2 k^2 \int_0^t F(\tau, k)(t - \tau) d\tau - \quad (4.2)$$
$$- c^2 \left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0} - 2i\,\omega k c^2 t \left(\frac{\partial f}{\partial y}\right)_{y=0} - c^2 \int_0^t \left(\frac{\partial B}{\partial y}\right)_{y=0} d\tau = 0$$

which reduces to an ordinary differential equation of the fourth order with constant coefficients, and the initial conditions are determined by the relation (4.2) and from the value of $A(t, y, k)$ at the point $t = y = 0$.

After the integral has been found, differentiation with respect to t leads to the function

$$F(t, k) = \frac{C_1 \cos 2\omega t + C_2 \sin 2\omega t + C_3 \cos kct + C_4 \sin kct}{\Delta(k, 2\omega / c)} +$$
$$+ \frac{1}{\Delta(k, 2\omega / c)} \int_0^t \left(\frac{\partial B}{\partial y}\right)_0 [\cos 2\omega(t - \tau) - \cos kc(t - \tau)] d\tau$$
$$\Delta(k, 2\omega / c) = k^2 - 4\omega^2 / c^2 \quad (4.3)$$

the constants of the fundamental solutions are determined by the characteristic values

$$C_1 = f''(0,\ k) - 4\omega^2 c^{-2} f(0,\ k),\quad C_2 = ikf'(0,\ k) - 2\omega c^{-2}\varphi'(0,k) \quad (4.4)$$
$$C_3 = k^2 f(0,k) - f''(0,k),\quad C_4 = kc^{-1}\varphi'(0,k) - 2i\omega c^{-1} f'(0,k)$$

Having found the function (4.3), our problem is solved and all that remains to round up our general analysis is to express explicitly the particular solution of (4.3) with zero initial conditions.

Outside the band $t \geqslant 0,\ 0 \leqslant y \leqslant l$ the Cauchy data (2.9) are continued in a periodic manner, and the boundary value of the normal derivative is calculated from the sum expression (3.6). The particular solution is then provided by a Duhamel-type integral

$$\frac{\overset{\circ}{q_1} + \overset{\circ}{q_2}}{c} \int_0^{ct} [\cos 2\omega\left(t - \frac{y}{c}\right) - \cos k(ct - y)] \left[\frac{1}{c} n(y, k) -\right.$$
$$\left. - \frac{s^2}{c^2} y\, m(y, k) + m'(y, k)\right] dy + \quad (4.5)$$
$$+ \frac{2}{c^2} \int_0^{ct} [\cos 2\omega\left(t - \frac{y}{c}\right) - \cos k(ct - y)] dy \sum_{q_1^\circ} \int_{2q_1^\circ l}^{y} [R'(\mu) n(\eta, k) +$$
$$+ 2cy R''(\mu) m(\eta, k)] \eta d\eta + \frac{2}{c^2} \int_0^{ct} [\cos 2\omega\left(t - \frac{y}{c}\right) - \cos k(ct - y)] dy \times$$

$$\times \sum_{q_1^\circ} \int_{(2q_2^\circ-1)l}^{y} [R'(\mu) n(\eta, k) + 2cy R''(\mu) m(\eta, k)] \eta \, d\eta \qquad \text{(cont.)}$$

The derivatives of the Riemann function are taken in the argument $\mu = y^2 - \eta^2$; the integers q_1°, q_2° follow from inequalities (3.5) considered at the boundary $y = 0$. The number of summands in the internal integrals of (4.5) differes at least by unity, hence a residual reflection wave may possibly be observed near a perpendicular wall. The function $F(t,k)$ defined by formulas (4.3)–(4.5), taken together, corresponds to the height of unsteady waves propagating along a perpendicular wall in a container. Expression (4.1) makes it possible to trace wave formation processes in the entire channel. The Fourier inversion serves to conclude our problem.

The author thanks L. N. Sretenskii for his advice on this paper.

BIBLIOGRAPHY

1. Sretenskii, L. N., A Hydrodynamic Problem Associated with the Tsunami Wave Problem. The Theory of Waves and Flow. Tr. morsk. gidrofiz. Inst., Vol. 27, 1963.

2. Hadamard, J. Résolution d'un problème aux limites pour les équations linéaires du type hyperbolique. Bull. Soc. Math. France, Vol. 32, Fsc. 3, pp. 242-268, 1904.

Translated by J. M. S.

SELF-SIMILAR SPECTRA OF DECAYING TURBULENCE AT LARGE REYNOLDS AND PÉCLET NUMBERS

PMM Vol. 33, №6, 1969, pp. 1091-1093

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(Received February 5, 1969)

We investigate the spectra of kinetic and thermal fluctuation energy of isotropic turbulence, neglecting the viscosity and molecular heat conduction. Elementary solutions of the corresponding spectral equations obtained here, enable us to investigate the properties of certain model spectra and a number of possible laws of decay in a simpler manner.

As we know [1], the spectral equations for the isotropic turbulence have the form

$$(\partial/\partial t + 2\nu k^2)\Phi(k, t) = -\partial/\partial k \; F(k, t) \tag{1}$$

$$(\partial/\partial t + 2\varkappa k^2)\; \Phi_{tt}(k, t) = -\partial/\partial k \; F_t(k, t) \tag{2}$$

where k is the wave number, t is time, F and F_t are the energy transfer functions, while Φ and Φ_{tt} are the fluctuation spectra defined by the equations

$$E(t) = \frac{1}{2}\langle u_i^2 \rangle = \int_0^\infty \Phi(k, t)\, dk \tag{3}$$

$$E_t(t) = \langle T'^2 \rangle = \int_0^\infty \Phi_{tt}(k, t)\, dk \tag{4}$$